$$\underline{x}(0) = \begin{bmatrix} -0.0306632 \\ -0.0567271 \\ -0.0788812 \\ -0.0977124 \\ -0.1137188 \\ -0.1273242 \end{bmatrix}$$

and the control vector is bounded such that

$$0 \leq u_1 \leq 1.0, \quad -0.4167 \leq u_2 \leq 0.972.$$

This system meets the requirements of the algorithms since

1. u_1 affects only $\dot{x}_1$ and u_2 affects only $\dot{x}_6$,
2. the normalized initial states are of the same sign,
3. and the x_i's between x_1 and x_6 cannot change sign as long as x_1 and x_6 do not change sign.

The application of the proposed algorithm is as follows. First at $t = 0$, the constrained value of the controls $u_1 = 1.0$ and $u_2 = 0.972$ are applied. At $t = 0.0586$, $x_1 = 0.000$ so u_1 is found from

$$u_1 = -(1/B_{1,1})\,[\underline{A}_1]\,\underline{x}$$

where $[A_i]$ represents the ith row of the $\underline{A}$ matrix. u_2 is kept at 0.972 until $t = 0.2025$. At that point $x_6 = 0.000$ and u_2 is found from

$$u_2 = -(1/B_{6,2})\,[\underline{A}_6]\,\underline{x}$$

From hence, u_1 and u_2 are calculated in a closed loop fashion until all the x_i are within a desired tolerance of the origin.

SUMMARY

An algorithm for generating the time-optimal deadbeat control policies for a class of lumped parameter systems has been presented and illustrated. The algorithm avoids the possibility of hidden oscillation, possible with other techniques proposed for solving this problem, because it is a continuous time algorithm. In addition, a portion of the algorithm is closed loop in nature, and only a small amount of computer storage is required to implement the solution.

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Viscosity of Saturated Nonpolar Liquids at Elevated Pressures

ATHENA LETSOU
Syracuse University, Syracuse, New York

and

LEONARD I. STIEL
University of Missouri, Columbia, Missouri

Numerous theoretical and experimental studies have been conducted on the viscosity of saturated liquids. Reid and Sherwood (1966) combined relationships of Thodos et al. (1962) for the dimensionless residual viscosity of dense gases and liquids as functions of reduced density with the equations of Stiel and Thodos (1961) for dilute nonpolar gases and the density correlation of Lydersen et al. (1955). The resulting procedure for the viscosity of saturated nonpolar liquids, valid between the normal boiling point and the critical point, is of the following functional form:

$$\mu\xi = f(T_R, z_c) \qquad (1)$$

where $\mu\xi R^{1/6}$ is dimensionless. In the present study, available viscosity data for a number of nonpolar liquids have been utilized to relate the dimensionless viscosity directly to reduced temperature and the equivalent third parameter for nonpolar fluids, the acentric factor, for reduced temperatures from 0.76 to 1.00.

In Figure 1 values of ln $\mu\xi$ are plotted against $1/T_R$ for n-butane ($\omega = 0.201$) and iso-butane ($\omega = 0.192$). It can be seen that at low temperatures to approximately $T_R = 0.7$, the saturated liquid viscosity follows the relationship (Reid and Sherwood, 1966)

$$\mu\xi = A\, e^{B/T_R} \qquad (2)$$

where A and B are constants for each substance which are determined primarily by the structure of the molecules. For isomers of n-pentane, n-hexane, and n-heptane, the slope of ln $\mu\xi$ versus $1/T_R$ also increases as the molecule becomes more spherical. There is no simple method to

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L. I. Stiel is with Allied Chemical Corporation, Specialty Chemicals Division, Buffalo, New York 14240.

calculate these constants, so that two experimental viscosities at two reduced temperatures in this region are required. There is a rough correlation (within approximately 20%) between $\mu\xi$ and T_R and ω for this region, with better results for the normal alkanes for which ω varies directly with the chain length of the molecules.

It can be seen from Figure 1 that from approximately $T_R = 0.76$ to $T_R = 1.0$, $\mu\xi$ follows the corresponding states relationship. There is undoubtedly some persistence of the structural contribution in this region, but the deviations from the generalized relationship are within the scatter of the experimental data for the various investigators. For the region of applicability of this approach the vapor pressure is substantially above one atmosphere, since the normal boiling point is between $T_R = 0.6$ and 0.7 for most nonpolar substances. The dependence of $\ln \mu\xi$ on T_R also follows Equation (2) at elevated pressures with constants A and B which vary with pressure (De Bock et al., 1967). Thus the values of these constants at high pressures are determined much more by molecular interactions than the low pressure values.

Experimental saturated liquid viscosity data for 14 normal fluids were utilized to establish values of $(\mu\xi)^{(0)}$ and $(\mu\xi)^{(1)}$ of the following relationship for T_R between 0.76 and 1.00 at 0.2 intervals:

$$\mu\xi = (\mu\xi)^{(0)}\,(T_R) + \omega\,(\mu\xi)^{(1)}\,(T_R) \qquad (3)$$

The sources and reduced temperature ranges of the data utilized are included in Table 1. It was found that the inclusion of a quadratic term in the least squares relationship did not markedly increase the accuracy. The resulting values of $(\mu\xi)^{(0)}$ and $(\mu\xi)^{(1)}$ for the T_R range considered are tabulated in Table 2.

The average percent error for interpolated values of $\mu\xi$ for the nonpolar fluids considered was 3.2% at $T_R = 0.76$, 3.5% at $T_R = 0.78$, and under 3.0% for T_R between 0.80 and 0.92. At high reduced temperatures the dependence of $\ln \mu\xi$ on $1/T_R$ is very steep, and the errors of the interpolated values of nitrogen and carbon dioxide were approximately 10% for $T_R = 0.94$ to 0.98. However, at high reduced temperatures, the dependence of $\mu\xi$ on ω becomes much smaller, as shown in Figures 2 to 4 in which $\mu\xi$ is plotted against reduced temperature at $T_R = 0.76$, 0.86, and 0.96, respectively. The relationship resulting from this study for the viscosity at the critical point

$$\mu_c\,\xi = 7.2 + 2.2\,\omega \qquad (4)$$

should be compared with the equation suggested by Hougen and Watson (1947)

$$\mu_c\,\xi = 7.7 \qquad (5)$$

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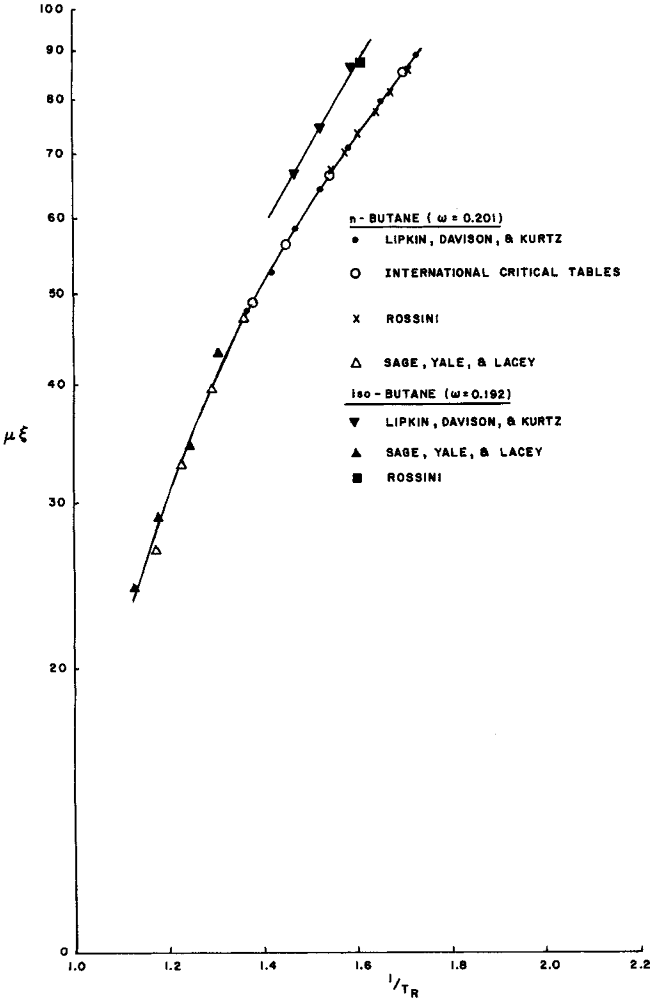

Fig. 1. Relationship between $\mu\xi$ and $1/T_R$ for *n*-butane and isobutane.

Table 1. Sources and Reduced Temperature Ranges of Data Included

Substance	Reference	T_R Range
Argon	Debock et al. (1967)	0.76-0.97
	Zhdanova (1957)	0.76-0.99
Nitrogen	Forster (1963)	0.76-0.96
	Rudenko (1939)	0.76-0.89
Methane	Gerf and Galkov (1941)	0.76-0.95
	Rudenko (1939)	0.70-0.88
	Swift et al. (1960)	0.76-1.00
Ethane	Carmichael and Sage (1963)	0.98-1.00
	Galkov and Gerf (1941)	0.76-0.94
	Gerf and Galkov (1940)	0.76-0.94
	Swift et al. (1960)	0.76-1.00
Ethylene	Gerf and Galkov (1941)	0.76-0.97
	Rudenko (1939)	0.76-1.00
Propane	Sage et al. (1964)	0.76-0.93
	Lipkin et al. (1942)	0.76-0.78
	Sage and Lacey (1938)	0.84-0.98
	Swift et al. (1959)	0.76-0.98
	Swift et al. (1960)	0.76-1.00
n-Butane	Sage et al. (1939)	0.76-0.85
	Swift et al. (1960)	0.76-0.88
iso-Butane	Sage et al. (1939)	0.76-0.88
Carbon Dioxide	Golubev (1959)	0.90-1.00
Benzene	Golubev (1959)	0.76-0.82
	Heiks and Orban (1956)	0.76-1.00
	Khalilov (1939)	0.76-0.86
n-Pentane	Agaev and Golubev (1963a)	0.76-1.00
	Albright and Lohrenz (1956)	0.88-0.92
	Hubbard and Brown (1943)	0.76-1.00
	Khalilov (1939)	0.76-0.90
n-Hexane	Albright and Lohrenz (1956)	0.78-0.90
	Khalilov (1939)	0.76-0.93
	Parisot and Johnson (1961)	0.76-0.90
n-Heptane	Agaev and Golubev (1963b)	0.76-1.00
	Albright and Lohrenz (1956)	0.79-0.84
	Khalilov (1939)	0.76-1.00
n-Octane	Agaev and Golubev (1963b)	0.76-1.00
	Albright and Lohrenz (1956)	0.76-0.79

Table 2. Values of Coefficients of Equation (3)

T_R	$(\mu\xi)^{(0)}$	$(\mu\xi)^{(1)}$
0.76	32.8	38.8
0.78	30.9	33.8
0.80	29.0	29.3
0.82	27.1	25.0
0.84	25.4	20.5
0.86	23.6	17.1
0.88	22.0	13.5
0.90	20.9	8.9
0.92	19.1	7.5
0.94	17.5	4.2
0.96	15.9	1.4
0.98	14.4	0.0
1.0	7.2	2.2

NOTATION

A, B = constants in Equation (2)
M = molecular weight
P_c = critical pressure, atm
R = gas constant, 82.05. cc atm/g-mole °K
T_c = critical temperature, °K
T_R = reduced temperature, T/T_c
z_c = critical compressibility factor

Greek Letters

μ = viscosity, micropoises
μ_c = viscosity at critical point, micropoises

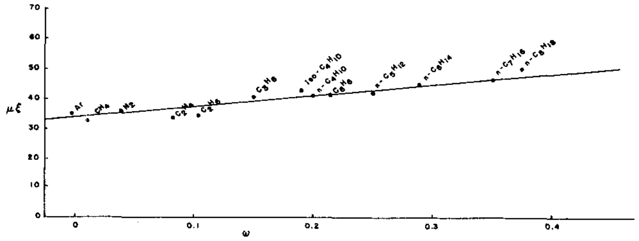

Fig. 2. Relationship between $\mu\xi$ and ω at $T_R = 0.76$.

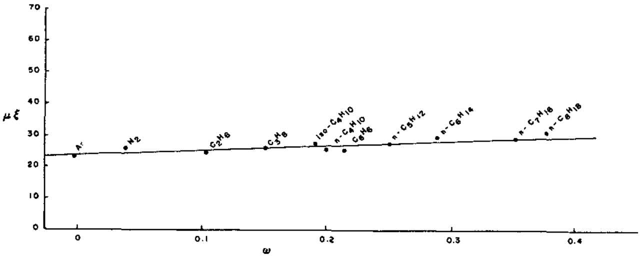

Fig. 3. Relationship between $\mu\xi$ and ω at $T_R = 0.86$.

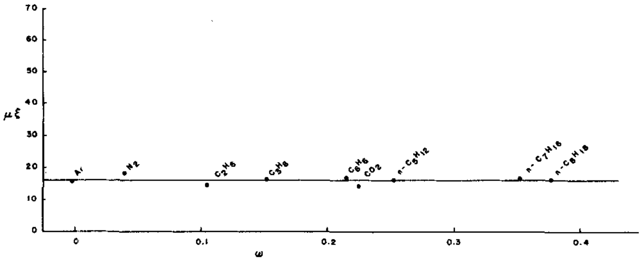

Fig. 4. Relationship between $\mu\xi$ and ω at $T_R = 0.96$.

ξ = $T_c^{1/6}/M^{1/2}P_c^{2/3}$
ω = acentric factor

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